A Model Error Formulation of the Multiple Model Adaptive Estimation Algorithm

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Abstract—This paper presents a new form of the multiple model adaptive estimation algorithm for improved state tracking in systems with unknown system models. The proposed approach differs from existing multiple model methods in the manner in which the covariance and Kalman gains of the individual filters are calculated. By using the fused model estimate, recursions for the actual estimation error covariances are derived which account for the deviation of the hypothesized model from the fused model. Using these covariances to determine the Kalman gain leads to improved tracking estimates through fusion of model and measurement uncertainty. The proposed algorithm has been compared against the standard multiple model adaptive estimation and interacting multiple model algorithms in two simulated examples, resulting in improved, and comparable tracking performance, respectively.

Index Terms—Multiple-model adaptive estimation, filtering, target tracking.

I. INTRODUCTION

Multiple model adaptive estimation (MMAE) is a recursive algorithm which uses a parallel bank of estimators (filters), each dependent upon a particular hypothesis, to determine a statistically rigorous estimate of the physical process under consideration. In particular, the hypotheses can correspond to different mathematical models of the same physical process, or of the same model but dependent upon different model parameters. The state estimate from an MMAE process is given by a weighted average of each filter's state estimate. The weights correspond to the normalized likelihood of the individual estimates conditioned on observed measurement sequence.

The concept of MMAE was first introduced by Magill for the problem of estimating the output from a plant in the presence of uncertain stochastic noise [1]. Since then, MMAE has found application is a wide variety of applications including parameter identification [2], control [3], fault detection [4], sensor calibration [5] and model identification [6]. The standard implantation of MMAE considers a finite set of hypotheses which are used to seed the bank of filters. The filters then independently run in parallel using their assigned hypothesis to reduce the available measurements. This approach typically works best when the hypothesized parameters or model are stationary or follow a prescribed trajectory defined by the hypothesis. In situations where the model parameters change, as is common in fault detection, a derivation of MMAE known as interacting multiple model (IMM) can yield improved performance over standard MMAE [7]. The switching of hypotheses in an IMM can be either random or deterministic, although the most common is to model the switches as a Markov sequence [11, 12]. Other variations of MMAE with time-varying hypothesis banks have also been developed [10], [11].

The driving force behind MMAE is the computation of the mode-conditioned likelihoods of the observed residual sequences which are used in calculating the conditional probabilities that each hypothesis correctly models the system. For observations corrupted by zero-mean Gaussian measurement noise, the mode-conditioned likelihood function take on the familiar form of a multivariate Gaussian probability density function with zero-mean and covariance $E^{(j)}$. The actual measurement-minus-estimate residual from the jth filter, $\mathbf{e}^{(j)}$, is used to evaluate the likelihood function and thus relating the conditional probability to the actually observed data. However, as discussed in Ref. [12], the covariance $E^{(j)}$ can represent the covariance of $\mathbf{e}^{(j)}$ if and only if the jth hypothesis is the true system model. Otherwise this covariance is nothing more than an approximation. In certain instances, when the hypothesis set does not include a suitable approximation to the true system, or in many nonlinear systems, such an approximation can lead to inconsistent (overconfident) estimates and /or divergence of the MMAE process. For linear systems, the actual innovations covariance can be recovered by accounting for the deviation of the modeled system from the true system [13]. For traditional filtering applications, such an approach is futile because the true system is necessarily unknown. However, via the multiple model approach, an estimate of the true system is available. This paper focuses on the fusion of these two tools into a single capability, called model error MMAE, which aims to improve overall tracking performance by leveraging MMAE-based estimates of the true system to drive the individual filters.

The organization of the remainder of this paper is as follows.

We begin with a terse review of the extended Kalman filter in order to introduce notion. Following this, the MMAE weight recursion is reviewed. Recursions for the actual estimation error covariance from a Kalman filter designed around an incorrect model are then presented. This derivation leads directly to the development of the model error MMAE. Two simulation examples are then presented in order to provide proof-of-concept and show the benefits of the proposed algorithm.

II. EXTENDED KALMAN FILTER

This work utilizes Kalman filters, including the extended Kalman filter (EKF) to reduce noise corrupted observations on an assumed dynamic model. This section provides a terse presentation of the well-known EKF so as to introduce notation for future developments. For more formal treatment of the Kalman filter and its derivation, interested readers are referred to other works such as Ref. [12]–[14].

Consider a nonlinear continuous-time system excited by noise:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t),\ t) + G(t)\,\mathbf{w}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the system state vector at time t, $\mathbf{f}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the system dynamic model, $G(t)$ is the process noise distribution matrix and $\mathbf{w}(t)$ is the process noise vector which is assumed to be a zero-mean Gaussian noise process with spectral density $Q(t)$., i.e. $\mathbf{w}(t) \sim \mathcal{N}\left(\mathbf{w}(t);\ \mathbf{0},\ Q(t)\right)$. Observations of the system state are available at discrete instances, k, and related to the system state as

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}_k) + \mathbf{v}_k \tag{2}$$

where $\mathbf{h}(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is the sensor observation model and $\mathbf{v}_k \sim \mathcal{N}\left(\mathbf{v}_k;\ \mathbf{0},\ R_k\right)$ is zero-mean Gaussian white noise which corrupts the observations. Realizations of the observations $\{\mathbf{y}_k\}$ are denoted as $\{\tilde{\mathbf{y}}_k\}$.

Let $\hat{\mathbf{x}}_k^+$ represent the *a posteriori* state estimate after processing all measurements, up to and including $\tilde{\mathbf{y}}_k$, and P_k^+ represent the covariance of the estimate errors. The prediction step, projecting the estimate forward in time is given by $\hat{\mathbf{x}}_{k+1}^- = \mathbf{f}_k(\hat{\mathbf{x}}_k^+)$ where the discrete nature of the propagation can result from either direct discretization of Eq. (1) or indirectly through numerical integration of the continuous-time dynamics. The predicted estimate, $\hat{\mathbf{x}}_{k+1}^-$ is the *a priori* state estimate at time $k + 1$. Oftentimes, if the sampling interval is below Nyquist's limit, a discrete-time propagation of the covariance is used:

$$P_{k+1}^- = \Phi_k P_k^+ \Phi_k^T + Q_k \tag{3}$$

where Φ_k is the discrete-time state transition matrix of $F(\hat{\mathbf{x}}(t), t) \equiv \left.\dfrac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}(t)}$ and Q_k is the discrete-time process noise covariance matrix. These matrices can be numerically computed for a constant sampling interval using an algorithm given by van Loan; details can be found in Ref. [14].

Upon receipt of a new measurement, $\tilde{\mathbf{y}}_k$, the estimate and covariance are updated:

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + K_k \mathbf{e}_k \tag{4a}$$

$$\mathbf{e}_k = \tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}_k^-) \tag{4b}$$

$$P_k^+ = [I - K_k H_k(\hat{\mathbf{x}}_k^-)]P_k^-[I - K_k H_k(\hat{\mathbf{x}}_k^-)]^T + K_k R_k K_k^T \tag{4c}$$

where $H_k(\hat{\mathbf{x}}_k^-) \equiv \left.\dfrac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\hat{\mathbf{x}}_k^-}$ and K_k is the Kalman gain:

$$K_k = P_k^- H_k^T(\hat{\mathbf{x}}_k^-) E_k^{-1} \tag{5a}$$

$$E_k \equiv \mathcal{E}\left[\mathbf{e}_k \mathbf{e}_k^T\right] = H_k(\hat{\mathbf{x}}_k^-)P_k^- H_k^T(\hat{\mathbf{x}}_k^-) + R_k \tag{5b}$$

where $\mathcal{E}[\cdot]$ is the expectation operator.

III. MULTIPLE MODEL ADAPTIVE ESTIMATION

This section provides a brief review of the multiple model adaptive estimation algorithm which serves to motivate the current work. A more thorough treatment of the subject can be found in Ref. [12]. Multiple model adaptive estimation (MMAE) is a recursive algorithm that uses a bank of filters operating in parallel where each filter is purposefully dependent upon a particular hypothesis, $\boldsymbol{\theta}^{(j)}$. A finite set of M hypothesis of $\boldsymbol{\theta}$ is available as $\boldsymbol{\Theta} = \left\{\boldsymbol{\theta}^{(1)}, \boldsymbol{\theta}^{(2)}, \ldots, \boldsymbol{\theta}^{(M)}\right\}$. This set can be comprised of random samples drawn from some *a priori* probability density function (pdf) $p(\boldsymbol{\theta})$, discretization of a continuous parameter space in $\boldsymbol{\theta}$, or an entire sample space in $\boldsymbol{\theta}$. The goal of the MMAE process is to determine $p(\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_{k+1}) \equiv p(\boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_{k+1})$, the conditional probability of the jth hypothesis, $\boldsymbol{\theta}^{(j)}$, given all available measurements:

$$p(\boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_{k+1}) = \frac{p(\boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_{k+1})}{p(\tilde{\mathbf{Y}}_{k+1})} \tag{6}$$

where $\tilde{\mathbf{Y}}_{k+1} = \{\tilde{\mathbf{y}}_{k+1}, \tilde{\mathbf{y}}_k, \ldots, \tilde{\mathbf{y}}_1\}$ is the sequence of measurements up to and including time index $k + 1$. Through repeated use of the definition of conditional probability and well as the law of total probability [15], Eq. (6) can be expressed as [12]

$$p\left(\boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_{k+1}\right) = \frac{p\left(\tilde{\mathbf{y}}_{k+1}|\tilde{\mathbf{Y}}_k, \boldsymbol{\theta}^{(j)}\right) p\left(\boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_k\right)}{\displaystyle\sum_{\ell=1}^{M}\left[p\left(\tilde{\mathbf{y}}_{k+1}|\tilde{\mathbf{Y}}_k, \boldsymbol{\theta}^{(\ell)}\right) p\left(\boldsymbol{\theta}^{(\ell)}|\tilde{\mathbf{Y}}_k\right)\right]} \tag{7}$$

The quantity $p(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k) = L(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k)$ is the mode-conditioned likelihood function at time index $k+1$. Also note that the denominator in Eq. (7) is simply a normalizing constant [12]. Defining $\varpi_k^{(\ell)} \equiv p(\boldsymbol{\theta}^{(j)}|\tilde{\mathbf{Y}}_k)$ as the weight associated with the jth hypothesis leads to the following recursion for the MMAE weights:

$$\varpi_{k+1}^{(j)} = \frac{\varpi_k^{(j)} L\left(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k\right)}{\displaystyle\sum_{\ell=1}^{M}\left[\varpi_k^{(\ell)} L\left(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta}^{(\ell)}, \tilde{\mathbf{Y}}_k\right)\right]} \tag{8}$$

When observations are corrupted by zero-mean Gaussian noise, such as in Eq. (2), the mode-conditioned likelihood function assumes the familiar form:

$$L\left(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k\right) \sim \mathcal{N}\left(\mathbf{e}_{k+1}^{(j)}; \mathbf{0}, E_{k+1}^{(j)}\right) \qquad (9)$$

The MMAE algorithm, by itself, provides only the probability that each hypothesis is correct to the system given the observation history. The state estimation results, however, can be interpreted in any manner appropriate to the system under consideration. For systems whereby the composition of the state vector is $\boldsymbol{\theta}$-dependent, a maximum *a posteriori* (MAP) interpretation of the results is adopted whereby the estimates from the mode-conditioned filter with the highest weight are adopted [5]. For systems with a $\boldsymbol{\theta}$-independent state vector composition, the system state estimate and its covariance are given by a weighted average of the individual estimates: [16]

$$\hat{\mathbf{x}}_k^+ = \sum_{j=1}^M \varpi_k^{(j)} \hat{\mathbf{x}}_k^{+(j)} \qquad (10a)$$

$$P_k^+ = \sum_{j=1}^M \varpi_k^{(j)} \left[P_k^{+(j)} + \left(\hat{\mathbf{x}}_k^{+(j)} - \hat{\mathbf{x}}_k^+\right)\left(\hat{\mathbf{x}}_k^{+(j)} - \hat{\mathbf{x}}_k^+\right)^T \right] \qquad (10b)$$

Estimates for the unknown model, $\hat{\boldsymbol{\theta}}$, are similarly available. Where appropriate, the $\hat{\boldsymbol{\theta}}_k$ can be accepted as the MAP hypothesis, or as a weighted average of all hypotheses:

$$\hat{\boldsymbol{\theta}}_k = \sum_{j=1}^M \varpi_k^{(j)} \boldsymbol{\theta}^{(j)} \qquad (11)$$

In many situations, interpretation of the estimated states and model will be treated similarly. That is, if the states estimates are obtained using Eq. (10a) then the model estimate will be determined using Eq. (11). However, it should be noted that a mixed interpretation is also valid. An example of such a mixed approach is shown is Ref. [6]. There the state estimates are obtained using Eq. (10a) while a MAP interpretation of the model is used.

Several results surrounding the convergence properties of MMAE have been presented in literature. For the case where the true hypothesis, *is* a member of the set of hypotheses, i.e. $\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)} \in \Theta$, Ref. [12] has shown that the true hypothesis will be identified with $p(\boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}|\tilde{\mathbf{Y}}_k) = 1$ as $k \to \infty$ while $p(\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}|\tilde{\mathbf{Y}}_k) = 0$, $i \neq j$, in the same interval. This proof assumes that the innovations sequence $\{\mathbf{e}_k^{(j)}\}$ is ergodic although the results are also valid when the innovations sequence is asymptotically wide sense stationary. Further, it was shown in Ref. [17] that convergence can also be shown in some non-stationary situations. When $\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)} \notin \Theta$, Ref. [12] shows that the MMAE algorithm will converge to $\boldsymbol{\theta}^{(j)}$, the member of the Θ which is closest to $\boldsymbol{\theta}$, in the sense of minimizing the Kullback information function [18].

IV. MODEL ERROR

As discussed in Ref. [12], the sequence $\{\mathbf{e}_k^{(j)}\}$ will have covariance given by $\{E_k^{(j)}\}$ if and only if $\boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}$, i.e.

if the acting hypothesis *is* the true model for the system. Even in the most ideal scenario, this will be true only for a single model in the set Θ. For all other $\boldsymbol{\theta}^{(i)} \neq \boldsymbol{\theta} \in \Theta$, it is necessarily the case that $\{E_k^{(i)}\}$ will not accurately represent the covariance of the observed sequence $\{\mathbf{e}_k^{(i)}\}$. When $\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)} \in \Theta$, such a result is trivial since the MMAE process will identify the correct model with $\varpi_k^{(j)} = 1$ and therefore Eq. (10) will result in consistent estimates of the system state and covariance. When $\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)} \notin \Theta$ the contrary also holds. Inexact in the knowledge of $\boldsymbol{\theta}$ will often lead to degraded filter performance and, consequently, degradation of the MMAE process. This is attributed to the fact that the filter is designed to a particular model. In specific, the errors manifest in the state error covariance matrix, and are distributed to the state estimates through the Kalman gain. In what follows, equations for the estimation error covariance are developed which account for the deviation of the hypothesized model from the actual system.

Let the true system under consideration be given by

$$\mathbf{x}_{k+1} = \mathbf{f}_k(\mathbf{x}_k, \boldsymbol{\theta}) + \Gamma_k \mathbf{w}_k \qquad (12a)$$

$$\mathbf{y}_k = H_k \mathbf{x}_k + \mathbf{v}_k \qquad (12b)$$

where the dynamic model of the system is dependent upon the model $\boldsymbol{\theta}$. As before, the discrete nature of Eq. (12a) can be intrinsic to the system or introduced as an approximation of the continuous-time system dynamics. Linearization of Eq. (12a) leads to

$$\mathbf{x}_{k+1} = \Phi_k(\boldsymbol{\theta})\mathbf{x}_k + \phi_k + \Gamma_k \mathbf{w}_k \qquad (13)$$

where $\Phi_k(\boldsymbol{\theta})$ is the time-dependent state transition matrix dependent upon the model $\boldsymbol{\theta}$ and, although not explicitly noted, the current state $\mathbf{x}_k$. A deterministic bias ϕ_k is present to account for the effects of nonlinearities [12]:

$$\phi_k \equiv \mathbf{f}_k(\mathbf{x}_k, \boldsymbol{\theta}) - \Phi_k(\boldsymbol{\theta})\mathbf{x}_k \qquad (14)$$

Suppose that $\boldsymbol{\theta}^{(j)} \neq \boldsymbol{\theta}$ is the model we wish to derive our filter around. It follows then that the linearized system to be designed to is modeled by [13]

$$\mathbf{x}_{k+1}^{(j)} = \Phi_k^{(j)} \mathbf{x}_k^{(j)} + \phi_k^{(j)} + \Gamma_k \mathbf{w}_k \qquad (15a)$$

$$\mathbf{y}_k^{(j)} = H_k \mathbf{x}_k^{(j)} + \mathbf{v}_k \qquad (15b)$$

where $\Phi_k^{(j)} = \Phi_k(\boldsymbol{\theta}^{(j)})$. The state error covariances resulting from a Kalman filter designed to Eq. (15) will necessarily not be consistent with the observed errors since the filter does not account for the model errors originating from the fact that the filter is designed to an incorrect model. To that extent, it is desired to develop expressions for the actual estimation error covariance, accounting for both measurement uncertainty and the modeling error.

Following [13], let the actual *a posteriori* and *a priori* estimation errors be defined as

$$\tilde{\mathbf{x}}_k^{+(j)} \equiv \mathbf{x}_k - \hat{\mathbf{x}}_k^{+(j)} \qquad (16a)$$

$$\tilde{\mathbf{x}}_{k+1}^{-(j)} \equiv \mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1}^{-(j)} \qquad (16b)$$

The observation model is assumed to be precisely known. The *a posteriori* errors therefore assume the familiar form [14]:

$$\tilde{\mathbf{x}}_k^{+(j)} = \left[I - K_k^{(j)} H_k\right] \tilde{\mathbf{x}}_k^{-(j)} - K_k^{(j)} \mathbf{v}_k \qquad (17)$$

The *a posteriori* covariance is then given by

$$\begin{aligned}
\mathcal{P}_k^{+(j)} &\equiv \mathcal{E}\left[\tilde{\mathbf{x}}_k^{+(j)} \tilde{\mathbf{x}}_k^{+(j)^T}\right] \\
&= \left[I - K_k^{(j)} H_k\right] \mathcal{P}_k^{-(j)} \left[I - K_k^{(j)} H_k\right]^T \\
&\quad + K_k^{(j)} R_k K_k^{(j)^T} \qquad (18)
\end{aligned}$$

where the *a posteriori* covariance has been defined as

$$\mathcal{P}_k^{-(j)} \equiv \mathcal{E}\left[\tilde{\mathbf{x}}_k^{-(j)} \tilde{\mathbf{x}}_k^{-(j)^T}\right] \qquad (19)$$

The *a priori* errors are given by substituting Eqs. (13) and (15a) into Eq. (16b), leading to

$$\tilde{\mathbf{x}}_{k+1}^{-(j)} = \Phi_k \mathbf{x}_k - \Phi_k^{(j)} \hat{\mathbf{x}}_k^{+(j)} + \Delta\phi_k^{(j)} + \Gamma_k \mathbf{w}_k \qquad (20)$$

where $\Delta\phi_k^{(j)} \equiv \phi_k - \phi_k^{(j)}$ and the explicit dependent of Φ_k on $\boldsymbol{\theta}$ has been omitted for clarity. Defining

$$\Delta\Phi_k^{(j)} \equiv \Phi_k - \Phi_k^{(j)} \qquad (21)$$

as the model error of the jth model allows Eq. (20) to be written as

$$\tilde{\mathbf{x}}_{k+1}^{-(j)} = \Phi_k^{(j)} \tilde{\mathbf{x}}_k^{+(j)} + \Delta\Phi_k^{(j)} \mathbf{x}_k + \Delta\phi_k^{(j)} + \Gamma_k \mathbf{w}_k \qquad (22)$$

Substituting this result into Eq. (19) leads to

$$\begin{aligned}
\mathcal{P}_{k+1}^{-(j)} &= \Phi_k^{(j)} \mathcal{P}_k^{+(j)} \Phi_k^{(j)^T} + \Phi_k^{(j)} \mathcal{E}\left[\tilde{\mathbf{x}}_k^{+(j)} \mathbf{x}_k^T\right] \Delta\Phi_k^{(j)^T} \\
&\quad + \Phi_k^{(j)} \mathcal{E}\left[\tilde{\mathbf{x}}_k^{+(j)}\right] \Delta\phi_k^{(j)^T} + \Delta\Phi_k^{(j)} \mathcal{E}\left[\mathbf{x}_k \tilde{\mathbf{x}}_k^{+(j)^T}\right] \Phi_k^{(j)^T} \\
&\quad + \Delta\Phi_k^{(j)} \mathcal{E}\left[\mathbf{x}_k \mathbf{x}_k^T\right] \Delta\Phi_k^{(j)^T} + \Delta\Phi_k^{(j)} \mathcal{E}\left[\mathbf{x}_k\right] \Delta\phi_k^{(j)^T} \\
&\quad + \Delta\phi_k^{(j)} \mathcal{E}\left[\tilde{\mathbf{x}}_k^{+(j)^T}\right] \Phi_k^{(j)^T} + \Delta\phi_k^{(j)} \mathcal{E}\left[\mathbf{x}_k^T\right] \Delta\Phi_k^{(j)^T} \\
&\quad + \Delta\phi_k^{(j)} \Delta\phi_k^{(j)^T} + \Gamma_k Q_k \Gamma_k^T \qquad (23)
\end{aligned}$$

After defining the following auxiliary quantities:

$$C_k^{+(j)} \equiv \mathcal{E}\left[\mathbf{x}_k \tilde{\mathbf{x}}_k^{+(j)^T}\right] \qquad (24a)$$

$$X_k = \mathcal{E}\left[\mathbf{x}_k \mathbf{x}_k^T\right] \qquad (24b)$$

$$\mathbf{m}_k = \mathcal{E}\left[\mathbf{x}_k\right] \qquad (24c)$$

$$\Delta\mathbf{m}_k^{+(j)} = \mathcal{E}\left[\tilde{\mathbf{x}}_k^{+(j)}\right] \qquad (24d)$$

Eq. (23) becomes

$$\begin{aligned}
\mathcal{P}_{k+1}^{-(j)} &= \Phi_k^{(j)} \mathcal{P}_k^{+(j)} \Phi_k^{(j)^T} + \Phi_k^{(j)} C_k^{+(j)^T} \Delta\Phi_k^{(j)^T} \\
&\quad + \Phi_k^{(j)} \Delta\mathbf{m}_k^{+(j)} \Delta\phi_k^{(j)^T} + \Delta\Phi_k^{(j)} C_k^{+(j)} \Phi_k^{(j)^T} \\
&\quad + \Delta\Phi_k^{(j)} X_k \Delta\Phi_k^{(j)^T} + \Delta\Phi_k^{(j)} \mathbf{m}_k \Delta\phi_k^{(j)^T} \\
&\quad + \Delta\phi_k^{(j)} \Delta\mathbf{m}_k^{+(j)^T} \Phi_k^{(j)^T} + \Delta\phi_k^{(j)} \mathbf{m}_k^T \Delta\Phi_k^{(j)^T} \\
&\quad + \Delta\phi_k^{(j)} \Delta\phi_k^{(j)^T} + \Gamma_k Q_k \Gamma_k^T \qquad (25)
\end{aligned}$$

Recursive relationships for the auxiliary equations are readily obtained using Eqs. (17), (20) and (24):

$$\begin{aligned}
X_{k+1} &= \Phi_k X_k \Phi_k^T + \Phi_k \mathbf{m}_k \phi_k^T + \phi_k \mathbf{m}_k^T \Phi_k^T \\
&\quad + \phi_k \phi_k^T + \Gamma_k Q_k \Gamma_k^T \qquad (26a)
\end{aligned}$$

$$\mathbf{m}_{k+1} = \Phi_k \mathbf{m}_k + \phi_k \qquad (26b)$$

$$\begin{aligned}
\Delta\mathbf{m}_{k+1}^{+(j)} &= \left[I - K_k^{(j)} H_k\right] \\
&\quad \times \left[\Phi_k^{(j)} \Delta\mathbf{m}_k^{+(j)} + \Delta\Phi_k^{(j)} \mathbf{m}_k + \Delta\phi_k^{(j)}\right] \qquad (26c)
\end{aligned}$$

The recursion for $C_k^{+(j)}$ is broken into two steps for simplicity. Substituting Eq. (17) into Eq. (24a) leads to

$$C_k^{+(j)} = C_k^{-(j)} \left[I - K_k^{(j)} H_k\right] \qquad (27)$$

where $C_k^{-(j)} \equiv \mathcal{E}\left[\mathbf{x}_k \tilde{\mathbf{x}}_k^{-(j)^T}\right]$. Using Eqs. (22) and (13) then leads to

$$\begin{aligned}
C_{k+1}^{-(j)} &= \Phi_k C_k^{+(j)} \Phi_k^{(j)^T} + \Phi_k X_k \Delta\Phi_k^{(j)^T} + \Phi_k \mathbf{m}_k \Delta\phi_k^{(j)^T} \\
&\quad + \phi_k \Delta\mathbf{m}_k^{+(j)^T} \Phi_k^{(j)^T} + \phi_k \mathbf{m}_k^T \Delta\Phi_k^{(j)^T} + \phi_k \Delta\phi_k^{(j)^T} \\
&\quad + \Gamma_k Q_k \Gamma_k^T \qquad (28)
\end{aligned}$$

A. Estimates of the True System

Generally speaking, the expressions for the actual estimation error covariances are not tractable since their computation requires knowledge of the true, yet unknown, state transition matrices $\{\Phi_k\}$, and biases $\{\phi_k\}$. For single filter applications, this means that the actual estimation error covariances cannot be determined. However, when combined with the multiple model approach of Section III, the actual estimation error covariances become computable quantities since estimates of the true system model are available. Using $\hat{\boldsymbol{\theta}}_k$ and $\hat{\mathbf{x}}_k^+$ from the MMAE algorithm it is possible to calculate the required state transition matrices and biases necessary to obtain estimates for the actual estimate error covariance.

B. Updated Kalman Gain Computation

The estimated actual estimation error can be used to provide more accurate bounds on the MMAE estimated state through Eq. (10b). However, this information is purely supervenient and will not impact the accuracy of the estimates themselves. However, this additional information can be incorporated into the individual filters by using the actual estimation error covariances to calculate the Kalman gain:

$$\mathcal{K}_k^{(j)} = \hat{\mathcal{P}}_k^{-(j)} H_k^T \left[H_k \hat{\mathcal{P}}_k^{-(j)} H_k^T + R_k\right]^{-1} \qquad (29)$$

A summary of the model error Kalman filter using the MMAE estimated quantities can be found in Table I. Note that the modifier $(\hat{\cdot})$, denoting an estimate, has been added to the actual covariance matrices $\hat{\mathcal{P}}_k^{+(j)}$ and $\hat{\mathcal{P}}_{k+1}^{-(j)}$ as well as the auxiliary variables to denote that these quantities are the result of using the MMAE estimated quantities. Additionally, herein $\hat{\mathcal{P}}_k^{+(j)}$ and $\hat{\mathcal{P}}_{k+1}^{-(j)}$ with be referred to as the *estimated actual estimation error covariances*.

Justification for using the Kalman gain in Eq. (29) can be seen as follows. During the update, the error is a difference of states generated by different systems. If the Kalman gain of Eq. (5a) is used, then no uncertainty due to the unknown model is ever introduced into the filter. As a result, the filter acts in a suboptimal manner, overconfident in the estimates. As a result, estimates will typically be inconsistent with the filter predicted covariance. In certain cases, the filter-produced covariance bounds may indicate a stable system while the actual estimation errors are unstable. By accounting for the uncertainty in the model, the estimated actual estimation error covariances are more conservative than $P_k^{+(j)}$ or $P_k^{-(j)}$. Therefore a Kalman gain derived using $P_k^{-(j)}$ will tend to rely on the model more than the observations which can lead to erroneous state estimates by inadvertently discounting the data derived from the actual system. By calculating the gain using Eq. (29), the filter will provide better balance between the uncertainties present in the model and observations.

C. MMAE using Model Error Kalman Filter

Since the sequence $\{E_k^{(j)}\}$ will be the the innovations sequence of $\{\mathbf{e}_k^{(j)}\}$ *if and only if* $\boldsymbol{\theta}^{(j)} = \boldsymbol{\theta}$, Eq. (9) can represent only an *approximation* to the mode-conditioned likelihood function because the observed innovations sequence and computed covariances are *not* matched, i.e. $\mathbf{e}_k^{(j)} = \boldsymbol{f}_1\left(\boldsymbol{\theta}, \boldsymbol{\theta}^{(j)}\right)$ while $E_k^{(j)} = \boldsymbol{f}_2\left(\boldsymbol{\theta}^{(j)}\right)$, only. By using the estimated actual estimate covariance, a better approximation to the covariance of $\mathbf{e}_{k+1}^{(j)}$ is available:

$$\mathbb{E}_{k+1}^{(j)} = H_{k+1}\hat{\mathcal{P}}_{k+1}^{-(j)}H_{k+1}^T + R_{k+1} \tag{30}$$

so that an alternate weight recursion to Eq. (8) using the estimated actual innovations covariance is given by

$$\omega_{k+1}^{(j)} = \frac{\omega_k^{(j)}\,\mathbb{L}(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k)}{\sum_{\ell=1}^{M}\left[\omega_k^{(\ell)}\,\mathbb{L}(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta} = \boldsymbol{\theta}^{(\ell)}, \tilde{\mathbf{Y}}_k)\right]} \tag{31}$$

with

$$\mathbb{L}(\tilde{\mathbf{y}}_{k+1}|\boldsymbol{\theta} = \boldsymbol{\theta}^{(j)}, \tilde{\mathbf{Y}}_k) \sim \mathcal{N}\left(\mathbf{e}_{k+1}^{(j)}; \mathbf{0}, \mathbb{E}_{k+1}^{(j)}\right) \tag{32}$$

This derivative of the MMAE algorithm utilizing Eq. (32) in conjunction with the Kalman filter in Table I will be herein referred to as the *model error MMAE* algorithm. Architecturally, model error MMAE differs from the standard MMAE in that each of the individual filters must have access to the fused MMAE output model and states. This architecture is similar in form to IMM. However, model error MMAE and IMM fundamentally differ in the information shared between the individual filters. With IMM, information regarding the state estimates and respective covariances are distributed whereas model error MMAE distributes knowledge about the fused model estimate. This also affords model error MMAE computational savings over IMM. The model error MMAE fusion of the best estimate model reduces to a fusion in $\boldsymbol{\theta}$ which will often be much smaller than the dimension of the state

space. Furthermore, IMM requires $2M$ fusions, M in the state vector and M in the covariance while model error MMAE requires only a single fusion. It should be noted that model error MMAE requires additional computations in maintaining the auxiliary quantities and determining the model errors. However, through simulation, it has been observed that model error MMAE still yields a computational savings over IMM.

V. NUMERICAL SIMULATIONS

A. Air Traffic Control Benchmark Problem

The first simulation considers a common example of a civilian air traffic control (ATC) tracking problem. In this problem, a target (aircraft) is tracked without *a priori* knowledge of any maneuvering the aircraft may perform. As discussed is Ref. [19], the three dimensional motion of the aircraft can typically be decoupled into the vertical and horizontal-planar motions with sufficient tracking results. For the current simulation, it is assumed that the aircraft is in level flight so that only the planar motion need be considered. While in level flight, an aircraft engages in only two modes of operation; straight-line motion and coordinated turns. The dynamics of this benchmark problem are well-known and therefore omitted here for brevity.

The target motion is tracked using standard MMAE, model error MMAE and IMM. The hypothesis set consists of nineteen coordinated turn models with different angular rates and a single constant velocity model. The nineteen angular rates are generated using a quasi-random Sobol sequence[1] on the support $[-5, 5]$. For the IMM algorithm, the transition probability matrix was defined so that the probability that a model did not transition was 10 times higher than the probability that a transition occurs. Since each model is not perfectly known, a small amount of process noise is assumed *by the filters only*, i.e. the true trajectory is not generated using any process noise disturbances. The affect of the process noise is characterized by [14]

$$\Gamma_k Q_k \Gamma_k^T = 0.05^2 \begin{bmatrix} \Delta t^3/3 & \Delta t^2/2 & 0 & 0 \\ \Delta t^2/2 & \Delta t & 0 & 0 \\ 0 & 0 & \Delta t^3/3 & \Delta t^2/2 \\ 0 & 0 & \Delta t^2/2 & \Delta t \end{bmatrix} \tag{33}$$

The initial covariance of the states are taken as 1000^2 m^2 in position and 100^2 (m/s)^2 in the velocity states.

As expected, standard MMAE is not able to track the target with any level of accuracy once the first maneuver is initiated. This is because by the time the first maneuver takes place, the algorithm has already given unity weight to the constant velocity model. Overall, model error MMAE and IMM provide accurate tracking performance with IMM providing slightly better performance near the maneuvers. The state estimation errors and 3σ covariance bounds can be found in Figure 1. As can be seen, the IMM slightly outperforms the model error MMAE although it should be noted that model error MMAE still provides for consistent estimates due to consideration of the estimated actual estimation error covariances.

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[1]http://www.mathworks.com/help/stats/sobolset.html, sobolset

Actual System	$\mathbf{x}_{k+1} = \Phi_k(\boldsymbol{\theta})\,\mathbf{x}_k + \phi_k + \Gamma_k\,\mathbf{w}_k, \qquad \mathbf{w}_k \sim \mathcal{N}(\mathbf{w}_k; \mathbf{0}, Q_k)$ $\mathbf{y}_k = H_k\,\mathbf{x}_k + \mathbf{v}_k, \qquad \mathbf{v}_k \sim \mathcal{N}(\mathbf{v}_k; \mathbf{0}, R_k)$
Modeled System	$\mathbf{x}_{k+1}^{(j)} = \Phi_k^{(j)}\,\mathbf{x}_k^{(j)} + \phi_k^{(j)} + \Gamma_k\,\mathbf{w}_k, \quad \Phi^{(j)} = \Phi(\boldsymbol{\theta}^{(j)})$ $\mathbf{y}_k^{(j)} = H_k\,\mathbf{x}_k^{(j)} + \mathbf{v}_k$
Initialize	$p(\mathbf{x}_0) = \mathcal{N}\left(\mathbf{x}_0; \hat{\mathbf{x}}_0^+, P_0^+\right), \qquad \hat{\mathbf{x}}_0^{+(j)} = \hat{\mathbf{x}}_0^+, \qquad \hat{\Phi}_0 = \Phi(\hat{\boldsymbol{\theta}}_0), \qquad \phi_0 = \phi(\hat{\mathbf{x}}_0^+)$ $\hat{\mathcal{P}}_0^{+(j)} = P_0^+, \qquad \hat{C}_0^{+(j)} = P_0^+, \qquad \hat{X}_0 = P_0^+ + \hat{\mathbf{x}}_0^+\,\hat{\mathbf{x}}_0^{+T}, \qquad \mathbf{m}_0 = \hat{\mathbf{x}}_0^+, \qquad \Delta\mathbf{m}_0^{+(j)} = \mathbf{0}$
Model Error	$\Delta\Phi_k^{(j)} \equiv \Phi(\hat{\boldsymbol{\theta}}_k) - \Phi_k^{(j)}, \qquad \Delta\phi_k^{(j)} \equiv \hat{\phi}_k - \phi_k^{(j)}$
Prediction	$\hat{\mathbf{x}}_{k+1}^{-(j)} = \Phi_k^{(j)}\hat{\mathbf{x}}_k^{+(j)} + \phi_k^{(j)}$ $\hat{\mathcal{P}}_{k+1}^{-(j)} = \Phi_k^{(j)}\hat{\mathcal{P}}_k^{+(j)}\Phi_k^{(j)T} + \Phi_k^{(j)}\hat{C}_k^{+(j)T}\Delta\Phi_k^{(j)T} + \Delta\Phi_k^{(j)}\hat{C}_k^{+(j)}\Phi_k^{(j)T} + \Phi_k^{(j)}\Delta\mathbf{m}_k^{+(j)}\Delta\phi_k^{(j)T} + \Delta\Phi_k^{(j)}\hat{X}_k\Delta\Phi_k^{(j)T}$ $+ \Delta\Phi_k^{(j)}\mathbf{m}_k\Delta\phi_k^{(j)T} + \Delta\phi_k^{(j)}\Delta\mathbf{m}_k^{+(j)T}\Phi_k^{(j)T} + \Delta\phi_k^{(j)}\mathbf{m}_k^T\Delta\Phi_k^{(j)T} + \Delta\phi_k^{(j)}\Delta\phi_k^{(j)T} + \Gamma_k Q_k \Gamma_k^T$ $\hat{C}_{k+1}^{-(j)} = \Phi(\hat{\boldsymbol{\theta}}_k)\hat{C}_k^{+(j)}\Phi_k^{(j)T} + \Phi(\hat{\boldsymbol{\theta}}_k)\hat{X}_k\Delta\Phi_k^{(j)T} + \Phi(\hat{\boldsymbol{\theta}}_k)\mathbf{m}_k\Delta\phi_k^{(j)T} + \hat{\phi}_k\Delta\mathbf{m}_k^{+(j)T}\Phi_k^{(j)T} + \hat{\phi}_k\mathbf{m}_k^T\Delta\Phi_k^{(j)T}$ $+ \hat{\phi}_k\Delta\phi_k^{(j)T} + \Gamma_k Q_k \Gamma_k^T$ $\hat{X}_{k+1} = \Phi(\hat{\boldsymbol{\theta}}_k)X_k\Phi(\hat{\boldsymbol{\theta}}_k)^T + \Phi(\hat{\boldsymbol{\theta}}_k)\mathbf{m}_k\hat{\phi}_k^T + \hat{\phi}_k\mathbf{m}_k^T\Phi(\hat{\boldsymbol{\theta}}_k)^T + \hat{\phi}_k\hat{\phi}_k^T + \Gamma_k Q_k \Gamma_k^T$ $\mathbf{m}_{k+1} = \Phi_k\mathbf{m}_k + \phi_k$ $\Delta\mathbf{m}_{k+1}^{+(j)} = \left[I - K_k^{(j)}H_k\right]\left[\Phi_k^{(j)}\Delta\mathbf{m}_k^{+(j)} + \Delta\Phi_k^{(j)}\mathbf{m}_k + \Delta\phi_k^{(j)}\right]$
Gain	$\mathcal{K}_k^{(j)} = \hat{\mathcal{P}}_k^{-(j)}H_k^T\left[H_k\hat{\mathcal{P}}_k^{-(j)}H_k^T + R_k\right]^{-1}$
Correction	$\hat{\mathbf{x}}_k^{+(j)} = \hat{\mathbf{x}}_k^{-(j)} + \mathcal{K}_k^{(j)}\left(\tilde{\mathbf{y}}_k - H_k\hat{\mathbf{x}}_k^{-(j)}\right)$ $\hat{\mathcal{P}}_k^{+(j)} = \left[I - \mathcal{K}_k^{(j)}H_k\right]\hat{\mathcal{P}}_k^{-(j)}\left[I - \mathcal{K}_k^{(j)}H_k\right]^T + \mathcal{K}_k^{(j)}R_k\mathcal{K}_k^{(j)T}$ $\hat{C}_k^{+(j)} = \hat{C}_k^{-(j)}\left[I - \mathcal{K}_k^{(j)}H_k\right]^T$

B. Space Object Tracking

A second example examines tracking of a space object in low Earth orbit with an unknown ballistic coefficient. For objects in low Earth orbit, atmospheric drag and the effect of Earth's oblateness constitute the primary perturbations from the two-body equations of motion. Truncating the effects of Earth gravity to include only the second zonal harmonic, the equations of motion are given by

$$\ddot{\mathbf{r}}(t) = \frac{\mu}{r^2}\mathbf{r}(t) + \mathbf{a}_{J_2} + \mathbf{a}_d \qquad (34)$$

where $\mathbf{a}_{J_2}$ is the perturbation due the second zonal harmonic and a function solely of the objects position and known gravitational parameters of the Earth [20]. The affect of atmospheric drag is given by

$$\mathbf{a}_d = -\frac{1}{2}\rho\,BC\,|\mathbf{v}_{\text{rel}}|^2\,\mathbf{u}_{\text{rel}} \qquad (35)$$

where ρ is the neutral density of the atmosphere in the vicinity of the space object, $BC = \frac{c_d A}{m}$ is the ballistic coefficient for the object. The nondimensional drag coefficient is given by c_d,

A is the affective area of the object in the direction of motion and m is the objects mass. The relative velocity of the object through the atmosphere, $\mathbf{v}_{\text{rel}}$ is assumed to be $\mathbf{v}_{\text{rel}} = \dot{\mathbf{r}} - \boldsymbol{\omega}_\oplus \times \mathbf{r}$ where $\boldsymbol{\omega}_\oplus$ is the angular rate of the Earth. Additionally $\mathbf{u}_{\text{rel}}$ is a unit vector in the direction of $\mathbf{v}_{\text{rel}}$.

The orbit of the object under consideration is given in terms of the Keplarian orbital elements as $\{a, e, i, \Omega, \omega\} = \{6775.741\text{km}, 0.0030035, 58.0579°, 54.0425°, 139.1568°\}$. Discrete dynamics in the form of Eq. (13) are attained through the linearization of Eq. (34) and subsequent discretization of the error dynamics [21]. Direct observations of the objects position, corrupted by zero-mean Gaussian white noise with a standard deviation of 25 m are assumed to be available at 10 second intervals. No process noise is assumed for this test case. The atmospheric model used is a time-invariant, exponential-decay model [22]. The true ballistic coefficient is given by $BC = 8.8$, which corresponds to an object with an average drag coefficient of 2.2 [22] and and A/m ratio of 4.0. This area-to-mass ratio is consistent with a High Area-to-Mass (HAMR) object [23] which are typically difficult to
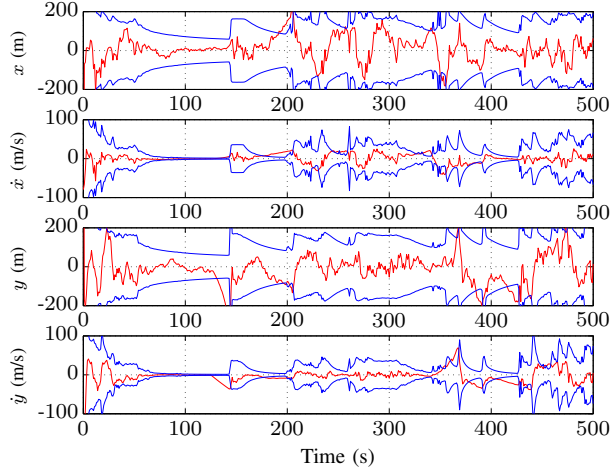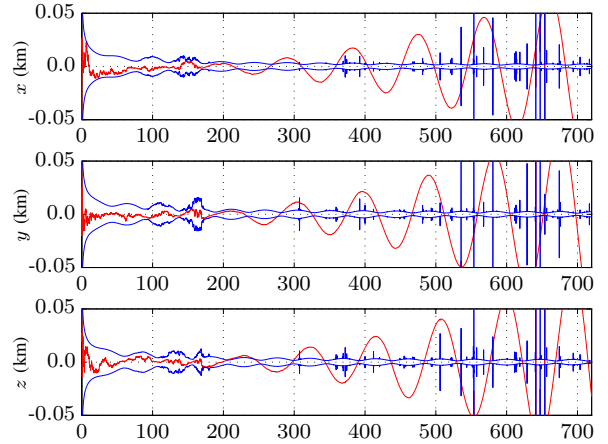
(a) Model Error MMAE
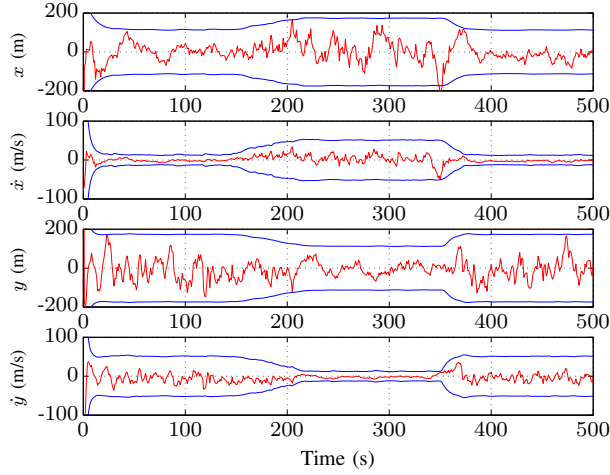
(b) IMM

Fig. 1. ATC Tracking Errors and 3σ Bounds

(a) MMAE

(b) Model Error MMAE

(c) IMM

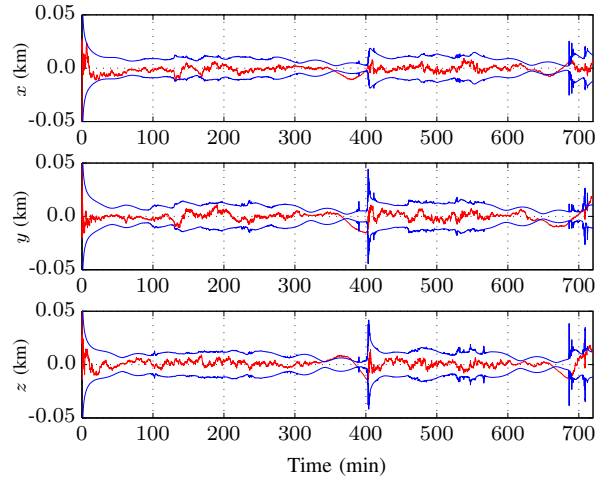Fig. 2. Space Object Position Tracking Errors and 3σ Bounds

track due to difficulties in predicting the perturbations such as drag.

The object is tracked using the MMAE, model error MMAE and IMM algorithms. The model bank consists of models with differing values for the ballistic coefficient. Ten of the models constitute standard space objects, with A/m values randomly selected from the uniform distribution on the unit interval. An additional ten models account for HAMR object with A/m selected from the uniform distribution with support $[1, 6]$.
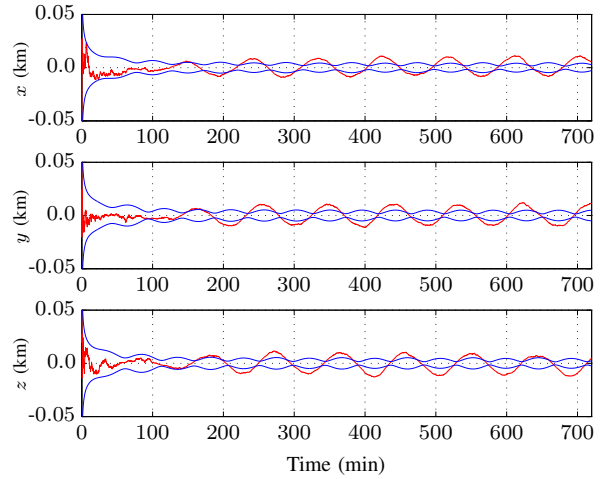
The position estimate errors for the three methods are shown in Figure 2. As can be seen, the MMAE estimates begin to quickly degrade. This is because the MMAE process converges upon a single model which is different from the actual ballistic coefficient. Then, by virtue of filtering with an incorrect system, the estimators are not able to effectively track the system. The model error MMAE is found to accurately track the system as shown in Figure 2(b). In particular, the estimates are found to be consistent with the predicted error bounds.

Consistency is obtained because the actual estimation error covariance, accounting for the model error are estimated. The influence of the model error augments the uncertainty as can be seen by comparing the respective 3σ bounds from the MMAE (Figure 2(a)) estimates with those from the model error MMAE. Further, since more accurate covariance information is available, the Kalman gain from Eq. (29) results in a more optimal balancing of prior and residual information, leading to improved tracking performance. The position estimation errors from the IMM can be found in Figure 2(c). Note that like the model error MMAE, the IMM appears able to sufficiently track the states although the errors are not consistent with the predicted covariances. From a practical perspective, this result may suffice but the inconsistency of the estimates, could prove troublesome if the tracking information was to be used in any subsequent analysis. It should be noted that, with regard to this particular example, that simultaneous estimation of the ballistic coefficient and states is possible through the use of an augmented system with an EKF or other suitable estimator [13] but that this approach was not taken so as to demonstrate feasibility of the proposed algorithm in comparison to other multiple model algorithms.

VI. CONCLUSIONS

This paper has presented a new method for state estimation using a multiple model adaptive estimator (MMAE). Called model error MMAE, the new estimator uses current knowledge of the fused model estimates in order to drive recursions for the actual estimation error covariances of the elementary filters. As such, the estimated actual estimation error covariances incorporate uncertainty accounting for the modeling errors of the hypothesized models. It was found that by using the estimated actual estimation error covariances to determine the Kalman gain led to improved tracking performance in two simulated examples. In both examples, the proposed model error MMAE outperformed the standard MMAE and was comparable to an interacting multiple model (IMM) estimator in tracking accuracy while producing more consistent estimates. Owing to the requirement for the fused model estimates, model error MMAE, like IMM, is more architecturally complex than standard MMAE and therefore cannot be operated in a truly decentralized manner. As compared to IMM, the proposed algorithm was also found to be computationally favorable. Additionally, the proposed algorithm alleviates the need for additional tuning parameters such as the transition probability matrix necessary with IMM. The developed approach is valid for linear systems or systems whereby the dynamics can be adequately modeled using a Taylor series truncated to first order. This paper has considered errors originating only in the dynamic model, although extensions to include errors in the observation model are straightforward. Further extensions to incorporate nonlinear filtering methods and other enhancements are the subject of currently ongoing research.

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